On the solution of linear and nonlinear integral equations based on spline quasi-interpolating projectors

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The problem

Integral equation

$$x - K(x) = f,$$

where:

- $K(x)(s) = \int_0^1 k(s,t)x(t)dt$, $s \in I = [0,1]$, $x \in C[0,1]$
  (Linear integral operator)
- $K(x)(s) = \int_0^1 k(s,t,x(t))dt$, $s \in I = [0,1]$, $x \in C[0,1]$
  (Urysohn integral operator)

The kernel k is a real valued continuous function and we assume that, for $f \in C[0,1]$, the integral equation has a unique solution φ.

Classical methods for integral equations

Standard technique

- Galerkin and collocation methods
- Kulkarni method (a modified projection method)
- Nyström method

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based on orthogonal projectors or interpolatory projectors onto finite dimensional subspaces X_n of $C[0,1]$.

Classical choice of $X_n \to$ space of piecewise polynomials of degree d at most continuous

[Atkinson 1973, 1992, 1997; Atkinson-Potra 1987; Kulkarni 2003, 2005; Grammont 2011; Grammont-Kulkarni 2009; Grammont-Kulkarni-Vasconcelos 2013; Allouch-Sbibih-Tahrichi 2014, 2017]

Spline quasi-interpolation for integral equations

- Quasi-interpolating splines of different degree for linear Fredholm integral equations

  [Allouch-Sablonnière-Sbibih 2011; Barrera-El Mokhtari-Sbibih 2018]

- Quasi-interpolating operators for 2D and surface integral equations

  [Allouch-Sablonnière-Sbibih 2013; Dagnino-Remogna 2017]

- Nyström method associated with non-uniform spline quasi-interpolation for Hammerstein integral equations

  [Barrera-El Mokhtari-Ibáñez-Sbibih 2018]

Our methods

Two methods based on spline quasi-interpolating projectors on the space of splines of degree d and smoothness C^{d-1}

Linear case [Dagnino-Remogna-Sablonnière 2014]

Nonlinear case [Dagnino-Dallefrate-Remogna 2018]

Outline

1. Spline quasi-interpolating projectors (QIPs)
2. Projection spline methods for linear and nonlinear integral equations
   - QIP spline Kulkarni's type method
   - QIP spline collocation method
3. Numerical tests

The spline space

Space of splines of degree d and class C^{d-1} on $\mathcal{T}_n$

$$\mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$$

- $\mathcal{T}_n = \{t_i = ih,\ 0 \le i \le n\}$
  uniform knot sequence with $h = 1/n$;
- $\mathcal{T}_n^e = \mathcal{T}_n \cup \{t_{-d} = \ldots = t_0 = 0; 1 = t_n = \ldots = t_{n+d}\}$
  extended knot sequence
- $N =$dim$(\mathcal{S}_d^{d-1}(I, \mathcal{T}_n)) = n + d$
- $\mathcal{B} = \{B_i,\ 1 \le i \le N\}$
  B-splines with support $[t_{i-d-1}, t_i]$, basis for $\mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$

Spline quasi-interpolating projectors

$$\pi_n : C[0,1] \to \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$$

$$\pi_n x = \sum_{i=1}^{N} \lambda_i(x) B_i,$$

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where $\{\lambda_i,\ 1 \leq i \leq N\}$ are local continuous linear functionals

$$\lambda_i(x) = \sum_{j=2(i-d-1)}^{2i} \sigma_{i,j} x\left(\xi_j\right),$$

with

- $\xi_{2i} = t_i$, for $0 \leq i \leq n$
- $\xi_{2i-1} = s_i = \frac{1}{2}(t_{i-1} + t_i)$, for $1 \leq i \leq n$
- $\sigma_{i,j}$ chosen such that $\pi_n x = x$, for all $x \in \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$

Spline quasi-interpolating projectors

$$\Downarrow$$

- the quasi-interpolation nodes ξ_j are inside the support of B_i
- π_n is a bounded projector, i.e. exact on $\mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$

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$$\Downarrow$$

$$\|x - \pi_n x\|_\infty \leq (1 + \|\pi_n\|_\infty)\, \mathrm{dist}(x, \mathcal{S}_d^{d-1}(I, \mathcal{T}_n))$$

$\Downarrow$ for $x \in C^j[0,1]$, $0 \leq j \leq d$

$$\|x - \pi_n x\|_\infty \leq \overline{C}_j h^j \omega(x^{(j)}, h)$$

$\Downarrow$ for $x \in C^{d+1}[0,1]$

$$\|x - \pi_n x\|_\infty = O(h^{d+1})$$

Properties of π_n

$$\pi_n x = \sum_{i=1}^{N} \lambda_i(x) B_i = \sum_{i=1}^{N} \sum_{j=2(i-d-1)}^{2i} \sigma_{i,j} x\left(\xi_j\right) B_i$$

Theorem 1

Let the degree d be even. If the functionals λ_i, $i = d+1, \ldots, n$, are such that the values $\sigma_{i,j}$ associated with quasi-interpolation nodes symmetric with respect to the center of the support of B_i, are equal, then

$$\int_{t_{i-1}}^{t_i} (\pi_n m_{d+1}(t) - m_{d+1}(t))dt = 0, \qquad i = d+1, \ldots, n-d,$$

where $m_{d+1}(t) = t^{d+1}$.

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Theorem 2

Let the degree d be even. If the functionals λ_i, $i = d+1, \ldots, n$, are such that the values $\sigma_{i,j}$ associated with quasi-interpolation nodes symmetric with respect to the center of the support of B_i, are equal, for any differentiable function g with $\|g'\|_1$ bounded and any function x such that $\left\|x^{(d+2)}\right\|_\infty$ is bounded, there results

$$\left| \int_0^1 g(t)(\pi_n x(t) - x(t)) dt \right| = O(h^{d+2}).$$

QIP spline Kulkarni's type method

Integral equation

$$\varphi - K(\varphi) = f,$$

K is approximated by $K_n^k = \pi_n K + K\pi_n - \pi_n K \pi_n$

$$\Downarrow$$

Approximate equation

$$\varphi_n^k - K_n^k(\varphi_n^k) = f$$

QIP spline Kulkarni's type method: convergence – Linear case

$\Omega = [0, 1] \times [0, 1]$

Theorem

For $\alpha \geq 1$, let $k \in C^{2\alpha}(\Omega)$, $f \in C^{2\alpha}[0, 1]$. Then

$$\|\varphi_n^k - \varphi\|_\infty = O(h^{2\beta}), \quad \beta = \min\{\alpha, d + 1\}.$$

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If the kernel of K is sufficiently smooth, that is $\alpha \geq d+1$

Theorem

For $\alpha \geq d+1$, let $k \in C^{2\alpha}(\Omega)$, $f \in C^{2\alpha}[0,1]$. Then

$$\|\varphi_n^k - \varphi\|_\infty = \begin{cases} O(h^{2d+2}), & \text{if } d \text{ is odd} \\ O(h^{2d+3}), & \text{if } d \text{ is even and } \pi_n \text{ satisfies} \\ & \text{the symmetry properties} \end{cases}$$

QIP spline Kulkarni's type method: convergence – Nonlinear case

Given $\alpha \geq 1$

- $\Omega = [0,1] \times [0,1] \times [a,b]$, $[\min_{s\in[0,1]} \varphi(s), \max_{s\in[0,1]} \varphi(s)] \subset (a,b)$
- $k \in C^{\alpha}(\Omega)$
- $\frac{\partial k}{\partial x} \in C^{2\alpha}(\Omega)$
- $f \in C^{\alpha}[0,1]$

K is a compact operator from $C[0,1]$ to $C^{\alpha}[0,1]$ and $\varphi \in C^{\alpha}[0,1]$.

K is Fréchet differentiable and the Fréchet derivative is given by

$$(K'(x)q)(s) = \int_0^1 \frac{\partial k}{\partial x}(s,t,x(t))q(t)dt.$$

QIP spline Kulkarni's type method: convergence – Nonlinear case

Theorem

For $\alpha \geq 1$, let $k \in C^{\alpha}(\Omega)$, $\frac{\partial k}{\partial x} \in C^{2\alpha}(\Omega)$ and $f \in C^{\alpha}[0,1]$. Assume that 1 is not an eigenvalue of $K'(\varphi)$. Then

$$\|\varphi_n^k - \varphi\|_\infty = O(h^{2\beta}), \quad \beta = \min\{\alpha, d+1\}.$$

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QIP spline Kulkarni's type method: solution construction – Linear case

Approximate solution

$$\varphi_n^k = f + \sum_{j=1}^{N} x_n(j) B_j + \sum_{j=1}^{N} y_n(j) K B_j, \quad x_n, y_n \in \mathbb{R}^N$$

we have to solve the following linear system of $2N$ equations

$$(I - D_n) z_n = d_n$$

$$D_n = \begin{bmatrix} C_n & B_n - C_n \\ I & C_n \end{bmatrix}, \quad z_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad d_n = \begin{bmatrix} v_n \\ w_n \end{bmatrix}$$

QIP spline Kulkarni's type method: solution construction – Linear case

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- $B_n \in \mathbb{R}^{N \times N}$

$$B_n(i,j) = \lambda_i \left(K^2 B_j\right)$$

- $C_n \in \mathbb{R}^{N \times N}$

$$C_n(i,j) = \lambda_i \left(K B_j\right)$$

- $v_n \in \mathbb{R}^N$

$$v_n(i) = \lambda_i(Kf)$$

- $w_n \in \mathbb{R}^N$

$$w_n(i) = \lambda_i(f)$$

QIP spline Kulkarni's type method: solution construction – Linear case

The system can be reduced to the solution of one system of N algebraic equations.

First we determine y_n by solving the linear system

$$((I - C_n)^2 + C_n - B_n)y_n = v_n + (I - C_n)w_n$$

then we get x_n by computing

$$x_n = (I - C_n)y_n - w_n$$

QIP spline Kulkarni's type method: solution construction – Nonlinear case

Define

$$F_n(y) = y - \pi_n K(y + (I - \pi_n)(K(y) + f)) - \pi_n f, \quad y \in \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$$

$$\Downarrow$$

we solve $F_n(\psi_n) = 0$, $\psi_n = \pi_n \varphi_n^k$ iteratively by Newton-Kantorovich method

QIP spline Kulkarni's type method: solution construction – Nonlinear case

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Given an initial approximation $\psi_n^{(0)}$, the iterates $\psi_n^{(r)}$, $r = 0, 1, 2, \ldots$, are

$$\begin{aligned}\psi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})\psi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})\psi_n^{(r+1)} \\ = \pi_n(K(\varphi_n^{(r)}) + f) - \pi_n K'(\varphi_n^{(r)})\psi_n^{(r)} \\ -\pi_n K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})\psi_n^{(r)}.\end{aligned}$$

with K' the Fréchet derivative of K and $\varphi_n^{(r)} = \psi_n^{(r)} + (I - \pi_n)(K(\psi_n^{(r)}) + f)$

QIP spline Kulkarni's type method: solution construction – Nonlinear case

Setting

$$\psi_n^{(r)} = \sum_{j=1}^{N} x_n^{(r)}(j) B_j, \quad x_n^{(r)} \in \mathbb{R}^N$$

we have to solve the following linear system of N equations

$$\left(I - A_n^{(r)} - B_n^{(r)}\right) x_n^{(r+1)} = d_n^{(r)}$$

- $A_n^{(r)} \in \mathbb{R}^{N \times N}$

$$A_n^{(r)}(i,j) = \lambda_i \left(K'(\varphi_n^{(r)}) B_j\right)$$

- $B_n^{(r)} \in \mathbb{R}^{N \times N}$

$$B_n^{(r)}(i,j) = \lambda_i \left(K'(\varphi_n^{(r)})(I - \pi_n) K'(\psi_n^{(r)}) B_j\right)$$

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we have to solve the following linear system of N equations

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- $d_n^{(r)} \in \mathbb{R}^N$

$$d_n^{(r)}(i) = \lambda_i \left(K(\varphi_n^{(r)})\right) + \lambda_i(f) - (A_n^{(r)} x_n^{(r)})(i) - (B_n^{(r)} x_n^{(r)})(i)$$

QIP spline Kulkarni's type method: solution construction – Nonlinear case

Approximate solution

$$\varphi_n^k = \varphi_n^{(r+1)} = \sum_{j=1}^{N} x_n^{(r+1)}(j)B_j + (I - \pi_n)\left(K\left(\sum_{j=1}^{N} x_n^{(r+1)}(j)B_j\right) + f\right)$$

QIP spline collocation method

Integral equation

$$\varphi - K(\varphi) = f,$$

- K is approximated by $K_n^c = \pi_n K \pi_n$
- f is approximated by $\pi_n f$

$$\Downarrow$$

Approximate equation

$$\varphi_n^c - \pi_n K \pi_n(\varphi_n^c) = \pi_n f$$

QIP spline collocation method: convergence

As expected from classical literature, we have

$$\|\varphi_n^c - \varphi\|_\infty = O(h^\beta),$$

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$$\Downarrow$$

$$\|\varphi_n^c - \varphi\|_\infty = O(h^{d+1})$$

QIP spline collocation method: solution construction – Linear case

Approximate solution

$$\varphi_n^c = \sum_{j=1}^{N} x_n(j) B_j, \quad x_n \in \mathbb{R}^N$$

we have to solve the following linear system of N equations

$$(I - C_n) x_n = w_n$$

- $C_n \in \mathbb{R}^{N \times N}$

$$C_n(i,j) = \lambda_i \left(K B_j \right)$$

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$$w_n(i) = \lambda_i(f)$$

QIP spline collocation method: solution construction – Nonlinear case

Define

$$F_n(y) = y - \pi_n K(y) - \pi_n f, \quad y \in \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$$

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Given an initial approximation $\varphi_n^{(0)}$, the iterates $\varphi_n^{(r)}$, $r = 0, 1, 2, \ldots,$ are

$$\varphi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})\varphi_n^{(r+1)} = \pi_n(K(\varphi_n^{(r)}) + f) - \pi_n K'(\varphi_n^{(r)})\varphi_n^{(r)},$$

with K' the Fréchet derivative of K.

QIP spline collocation method: solution construction – Nonlinear case

Setting

$$\varphi_n^{(r)} = \sum_{j=1}^{N} x_n^{(r)}(j) B_j, \quad x_n^{(r)} \in \mathbb{R}^N$$

we have to solve the following linear system of N equations

$$(I - C_n^{(r)}) x_n^{(r+1)} = w_n^{(r)}$$

- $C_n^{(r)} \in \mathbb{R}^{N \times N}$

$$C_n^{(r)}(i,j) = \lambda_i \left(K'(\varphi_n^{(r)}) B_j \right)$$

- $w_n^{(r)} \in \mathbb{R}^N$

$$w_n^{(r)}(i) = \lambda_i \left(K(\varphi_n^{(r)}) \right) + \lambda_i(f) - (C_n^{(r)} x_n^{(r)})(i)$$

QIP spline collocation method: solution construction – Nonlinear case

Approximate solution

$$\varphi_n^c = \varphi_n^{(r+1)} = \sum_{j=1}^{N} x_n^{(r+1)}(j) B_j, \quad x_n^{(r)} \in \mathbb{R}^N$$

Numerical tests

- We consider the QIP Q_d, of degree $d = 2, 3$ proposed in [Dagnino-Remogna-Sablonnière 2014]

$$Q_2 x = \sum_{i=1}^{n+2} \lambda_i(x) B_i, \text{ with}$$

$$\lambda_1(x) = x_0, \qquad \lambda_2(x) = 2x_1 - \tfrac{1}{2}(x_0 + x_2),$$

$$\lambda_i(x) = \tfrac{1}{14} x_{2i-6} - \tfrac{2}{7} x_{2i-5} + \tfrac{10}{7} x_{2i-3} - \tfrac{2}{7} x_{2i-1} + \tfrac{1}{14} x_{2i}, \; 3 \leq i \leq n,$$

$$\lambda_{n+1}(x) = 2x_{2n-1} - \tfrac{1}{2}(x_{2n-2} + x_{2n}), \qquad \lambda_{n+2}(x) = x_{2n}.$$

Numerical tests

$$Q_3 x = \sum_{i=1}^{n+3} \lambda_i(x) B_i, \text{ with}$$

$$\lambda_1(x) = x_0,$$

$$\lambda_2(x) = -\frac{5}{18}x_0 + \frac{20}{9}x_1 - \frac{4}{3}x_2 + \frac{4}{9}x_3 - \frac{1}{18}x_4,$$

$$\lambda_3(x) = \frac{1}{8}x_0 - x_1 + \frac{19}{8}x_2 - \frac{19}{24}x_4 + \frac{1}{3}x_5 - \frac{1}{24}x_6,$$

$$\lambda_i(x) = -\frac{1}{30}x_{2i-8} + \frac{4}{15}x_{2i-7} - \frac{19}{30}x_{2i-6} + \frac{9}{5}x_{2i-4} - \frac{19}{30}x_{2i-2} + \frac{4}{15}x_{2i-1} - \frac{1}{30}x_{2i}, \quad 4 \leq i \leq n,$$

$$\lambda_{n+1}(x) = \frac{1}{8}x_{2n} - x_{2n-1} + \frac{19}{8}x_{2n-2} - \frac{19}{24}x_{2n-4} + \frac{1}{3}x_{2n-5} - \frac{1}{24}x_{2n-6},$$

$$\lambda_{n+2}(x) = -\frac{5}{18}x_{2n} + \frac{20}{9}x_{2n-1} - \frac{4}{3}x_{2n-2} + \frac{4}{9}x_{2n-3} - \frac{1}{18}x_{2n-4},$$

$$\lambda_{n+3}(x) = x_{2n}.$$

Numerical tests

- Q_2 is superconvergent on the set of quasi-interpolation nodes $\{\xi_i\}_{i=0}^{2n}$.
  If $\|x^{(4)}\|_\infty$ is bounded, then

$$|Q_2 x(\xi_i) - x(\xi_i)| = O(h^4), \qquad 0 \leq i \leq 2n.$$

- The integrals appearing in the linear systems are evaluated numerically by using a classical composite Gauss-Legendre quadrature formula with high accuracy

Numerical tests

- For increasing values of n, we compute:
  i) the maximum absolute error on a set G of 1500 equally spaced points in $[0, 1]$

$$E_\infty^\mu = \max_{v \in G} |\varphi(v) - \varphi_n^\mu(v)|, \quad \mu = c, k$$

  O_∞^μ: corresponding numerical convergence order

  ii) the maximum absolute error at the quasi-interpolation nodes

$$ES^\mu = \max_{0 \le i \le 2n} |\varphi(\xi_i) - \varphi_n^\mu(\xi_i)|, \quad \mu = c, k$$

  O^μ: corresponding numerical convergence order

Test 1 – Linear integral equation

$$\varphi(s) - \int_0^1 e^{st}\varphi(t)dt = e^{-s}\cos(s),$$

with $\varphi(s) = e^{-s}\cos(s)$, $s \in [0,1]$.

n	E^k_∞	O^k_∞	ES^k	O^k	E^c_∞	O^c_∞	ES^c	O^c
				Methods based on Q_2				
4	1.08(-09)		1.62(-10)		2.66(-04)		1.28(-04)	
8	6.32(-12)	7.4	4.49(-13)	8.5	3.08(-05)	3.1	6.51(-06)	4.3
16	4.11(-14)	7.3	2.22(-15)	7.7	3.89(-06)	3.0	4.87(-07)	3.7
32	-	-	-	-	4.89(-07)	3.0	3.29(-08)	3.9
64	-	-	-	-	6.07(-08)	3.0	2.13(-09)	3.9
128	-	-	-	-	7.59(-09)	3.0	1.35(-10)	4.0
				Methods based on Q_3				
4	1.58(-11)				3.27(-05)			
8	3.30(-14)	9.0			2.25(-06)	3.9		
16	-	-			1.47(-07)	3.9		
32	-	-			9.37(-09)	4.0		
64	-	-			5.92(-10)	4.0		
128	-	-			3.68(-11)	4.0		

Test 2 – Nonlinear integral equation of Hammerstein type

$$\varphi(s) - \int_0^1 \cos(11\pi s)\sin(11\pi t)\varphi^2(t)dt = \left(1 - \frac{2}{33\pi}\right)\cos(11\pi s),$$

with $\varphi(s) = \cos(11\pi s)$, $s \in [0, 1]$.

n	E^k_∞	O^k_∞	ES^k	O^k	E^c_∞	O^c_∞	ES^c	O^c
Methods based on Q_2								
40	1.08(-06)		6.97(-07)		7.74(-03)		4.98(-03)	
80	4.08(-09)	8.1	2.26(-09)	8.2	6.77(-04)	3.5	3.76(-04)	3.7
160	2.13(-11)	7.6	6.31(-12)	8.5	8.17(-05)	3.0	2.43(-05)	4.0
320	1.42(-13)	7.2	2.14(-14)	8.2	1.01(-05)	3.0	1.53(-06)	4.0
640	-	-	-	-	1.26(-06)	3.0	9.57(-08)	4.0
Methods based on Q_3								
40	2.38(-08)				1.53(-03)			
80	9.40(-11)	8			9.27(-05)	4.0		
160	1.12(-13)	9.7			5.58(-06)	4.1		
320	-	-			3.43(-07)	4.0		
640	-	-			1.34(-08)	4.7		

Test 2 – Nonlinear integral equation of Uryshon type

$$\varphi(s) - \int_0^1 \frac{dt}{s + t + \varphi(t)} = f(s), \quad s \in [0,1],$$

f chosen so that $\varphi(s) = \dfrac{1}{s+c}$, $c > 0$, is a solution.

$$c = 1$$

n	E^k_∞	O^k_∞	ES^k	O^k	E^c_∞	O^c_∞	ES^c	O^c
Methods based on Q_2								
4	8.48(-08)		5.06(-08)		6.85(-04)		3.84(-04)	
8	7.84(-10)	6.8	3.50(-10)	7.2	9.54(-05)	2.8	3.84(-05)	3.3
16	5.08(-12)	7.3	1.47(-12)	7.9	1.21(-05)	3.0	3.10(-06)	3.6
32	3.08(-14)	7.4	5.55(-15)	8.0	1.50(-06)	3.0	2.21(-07)	3.8
64	-	-	-	-	1.85(-07)	3.0	1.48(-08)	3.0
Methods based on Q_3								
4	1.58(-09)				9.02(-05)			
8	3.30(-12)	8.9			7.77(-06)	3.5		
16	6.55(-15)	9.0			6.84(-07)	3.5		
32	-	-			5.00(-08)	3.8		
64	-	-			3.36(-09)	3.9		

Test 2 – Nonlinear integral equation of Uryshon type

$$\varphi(s) - \int_0^1 \frac{dt}{s+t+\varphi(t)} = f(s), \quad s \in [0,1],$$

f chosen so that $\varphi(s) = \dfrac{1}{s+c}$, $c > 0$, is a solution.

$$c = 0.1$$

n	E^k_∞	O^k_∞	ES^k	O^k	E^c_∞	O^c_∞	ES^c	O^c
Methods based on Q_2								
4	4.50(-07)		1.13(-07)		6.80(-01)		5.51(-01)	
8	3.87(-10)	10.2	2.51(-10)	8.8	2.67(-01)	1.3	2.10(-01)	1.4
16	1.00(-11)	5.3	1.01(-12)	8.0	7.04(-02)	1.9	5.12(-02)	2.0
32	1.21(-13)	6.4	1.07(-14)	6.7	1.29(-02)	2.4	7.99(-03)	2.7
64	-	-	-	-	1.86(-03)	2.8	8.65(-04)	3.2
Methods based on Q_3								
4	1.97(-08)				4.17(-01)			
8	1.16(-11)	10.7			1.03(-01)	2.0		
16	1.29(-13)	6.5			1.70(-02)	2.6		
32	-	-			1.96(-03)	3.1		
64	-	-			1.75(-04)	3.5		

Work in progress

- Not sufficiently smooth kernels

Work in progress

- Not sufficiently smooth kernels

Work in progress

- Not sufficiently smooth kernels
- Spline quasi-interpolating operators that are not projectors

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Thank you!