Hermite-Birkhoff spline quasi-interpolation descending from a class of Hermite-Obrechkoff schemes for ODEs

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Recent advances in Quasi–Interpolation and Applications

High order methods for 1D Initial Value Problems

$$IVP \begin{cases} \mathbf{y}'(t) & = \mathbf{f}(t, \mathbf{y}(t)) \;\; t \in [t_0\,,\; t_0 + T], \\ \mathbf{y}(t_0) & = \mathbf{y}_0 \end{cases}$$

with $\mathbf{f} : [t_0\,,\; t_0 + T] \times \mathbb{R}^m \to \mathbb{R}^m$, $\mathbf{y} : [t_0\,,\; t_0 + T] \to \mathbb{R}^m$.

Types of methods:

- one-step methods with multiple stages (Runge-Kutta);
- multiderivative one-step methods;
- multistep methods (also multiderivative).

We consider Hermite-Obrechkoff (HO) methods of the second type,

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \sum_{j=1}^{R} h_n^j \, \beta_j^{(R)} \left(\mathbf{u}_n^{(j)} - (-1)^j \mathbf{u}_{n+1}^{(j)} \right), \; n = 0, \ldots, N-1.$$

$$\mathbf{u}_n^{(j)} \approx \mathbf{y}^{(j)}(t_n) = \frac{d^{j-1}}{d^{j-1}t} \mathbf{f}(t_n, \mathbf{y}(t_n)), \qquad t_n = t_{n-1} + h_n.$$

EMHO and BSHO methods

- EMHO: Euler-Maclaurin HOs, order $2s$

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \frac{h}{2}\left(\mathbf{u}_n^{(1)} + \mathbf{u}_{n+1}^{(1)}\right) + \sum_{i=1}^{s-1} h_n^{2i}\,\frac{b_{2i}}{(2i)!}\left(\mathbf{u}_n^{(2i)} - \mathbf{u}_{n+1}^{(2i)}\right),$$

  where the b_k denote the Bernoulli numbers.

- BSHO: B-Spline HOs, order $2R$
  (same order with lower order derivatives)

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \sum_{j=1}^{R} h_n^j\,\beta_j^{(R)}\left(\mathbf{u}_n^{(j)} - (-1)^j\mathbf{u}_{n+1}^{(j)}\right)$$

where $\beta_j^{(R)} := \frac{1}{j!}\frac{R(R-1)\dots(R-j+1)}{(2R)(2R-1)\dots(2R-j+1)}$.

BSHOs

First introduction of BSHOs :

F. Loscalzo, PhD Thesis, 1968

y polynomial of degree $\leq D := 2R \Rightarrow$ **y** fulfills the BSHO recurrence

$\Downarrow$

R–multiple collocation at the mesh points of a D–degree spline
minimal assumption on the spline regularity: C^R

Recently we have reconsidered BSHOs:

- their conjugate symplecticity has been proved (useful for application to conservative problems);
- an efficient technique for the computation of the B-spline coefficients of the associated C^R collocating spline has been introduced.

F. Mazzia and AS, preprint.

Generalization of the spline definition in the QI context

- Values $\mathbf{y}_n^{(j)}$, $j \leq R$, from BSHO $\Rightarrow$ $2R$–degree collocating spline;
- Values $\mathbf{y}_n^{(j)}$, $j \leq R$, from other HO $\Rightarrow$ D–degree QI spline.

$S_D = D$–degree spline space

B-spline basis for the C^R case: $B_i, i = -D, \ldots, (N-1)R$

- breakpoints at the mesh points: $t_0 < \ldots < t_N$;
- $C^R \Rightarrow$ R–multiple inner knots in the extended knot vector Θ :

$$\Theta := \{\theta_{-2R}, \ldots, \theta_{-1}, \theta_0, \ldots, \theta_{(N-1)R}, \theta_{(N-1)R+1}, \theta_{(N-1)R+2} \cdots, \theta_{(N+1)R+1}\}$$

where

$$\begin{array}{llll} \theta_{-2R} & = \cdots = \theta_0 := t_0, & \theta_{(N-1)R+1} & = \cdots = \theta_{(N+1)R+1} := t_N, \\ \theta_{(n-1)R+1} & = \cdots = \theta_{nR} := t_n, & & n = 1, \ldots, N-1. \end{array}$$

- $B_i \neq 0$ only in two or at most three consecutive mesh cells.

C^R spline definition

$$\sigma(\cdot) := \sum_{i=-D}^{(N-1)R} \lambda_i(y) B_i(\cdot)$$

- $\lambda_i(y) =$ linear combination of the values $y_{n_i}^{(j)}, \, y_{n_i+1}^{(j)}, \, j \le R$, with
- $(t_{n_i} \, , \, t_{n_i+1}) =$ central (if exists) or left subinterval included in $\mathrm{supp}(B_i), 0 \le n_i \le N-1$
  (almost forward; fully forward variant also usable);
- $\Rightarrow$ high locality of the scheme.

Coefficients computation:

- $[t_k \, , \, t_{k+1}] \leftrightarrow D$–degree polynomial P_k and an additional constant τ_k
- P_k and τ_k are computed by solving a nonsingular local linear system of size $(2R+2) \times (2R+2)$
- the computational cost is linear in N; explicit formulas available for uniform mesh distributions.

Spline coefficients

$$P_k^{(j)}(t_k) = y_k^{(j)}\,, \qquad P_k^{(j)}(t_{k+1}) = y_{k+1}^{(j)}\,, \quad j \neq 1$$

$$P_k^{(1)}(t_k) = y_k^{(1)} + \tau_k/h_k \quad P_k^{(1)}(t_{k+1}) = y_{k+1}^{(1)} + \tau_k/h_k\,.$$

Expressing P_k as a linear combination of B-splines active in $[t_k\,,\, t_{k+1}]$,

$$P_k = (B_{(k-2)R}, \cdots, B_{kR})\mathbf{c}^{(k)}\,, \quad \mathbf{c}^{(k)} \in \mathbb{R}^{2R+1}\,,$$

$$\Updownarrow$$

$$G^{(k)T}(\mathbf{c}^{(k)T}\,,\, \tau_k)^T = \mathbf{w}^{(k)}(y) \quad \text{squared linear system}$$

$$\mathbf{w}^{(k)}(y) := (y_k^{(0)}, y_{k+1}^{(0)}, -h_k y_k^{(1)}, -h_k y_{k+1}^{(1)}, \cdots, -h_k^R y_k^{(R)}, -h_k^R y_{k+1}^{(R)},)^T$$

Local coefficient matrix

$$G^{(k)} := \begin{bmatrix} A_1^{(k)T} & -h_k A_2^{(k)T} & -h_k^2 A_3^{(k)T} & \dots & -h_k^R A_{R+1}^{(k)T} \\ \mathbf{0}^T & \mathbf{e}^T & \mathbf{0}^T & \dots & \mathbf{0}^T \end{bmatrix}_{(2R+2)^2}$$

$$A_{j+1}^{(k)} := \begin{bmatrix} B_{(k-2)R}^{(j)}(t_k), & \dots, & B_{kR}^{(j)}(t_k) \\ B_{(k-2)R}^{(j)}(t_{k+1}), & \dots, & B_{kR}^{(j)}(t_{k+1}) \end{bmatrix}_{2\times(2R+1)} \qquad j = 0, \dots, R$$

$\Rightarrow$ σ definition in the B-spline basis:

$B_i \leftrightarrow \lambda_i(u) =$ $(i-(n_i-2)R+1)$–th entry of $\mathbf{c}^{(n_i)} \leftrightarrow P_{n_i}$

REMARK: τ_k is the k–th local truncation error of the BSHO method for y $\Rightarrow$ it vanishes if $y \in S_D \Rightarrow$ the considered spline QI scheme is a projector; it is $O(h_k^{D+1})$ if y is sufficiently smooth.

Uniform BSHO-QI coefficients

D=2, R=1

quasi-interpolant associated to the trapezoidal rule

$$\lambda_{-2} = y_0,$$
$$\lambda_{i-2} = \frac{h}{2}(y_{i+1} + y_i) - \frac{h}{4}(y'_{i+1} - y'_i), i = 1, \cdots, N$$
$$\lambda_{N-1} = y_N,$$

D=4, R=2

quasi-interpolant associated to the Euler–Maclaurin method of order 4

$$\lambda_{-4} = y_0,$$
$$\lambda_{-3} = \frac{1}{4}(y_{i+1} + 3y_i) - \frac{h}{8}(y'_{i+1} - y'_i) + \frac{h^2}{48}(y''_{i+1} - y''_i),$$
$$\lambda_{2i-4} = \frac{1}{2}(y_{i+1} + y_i) - \frac{h}{4}(y'_{i+1} - y'_i) + \frac{h^2}{24}(y''_{i+1} + y''_i), i = 1, \cdots, N$$
$$\lambda_{2i-3} = y_{i+1} - \frac{h^2}{12}y''_{i+1}, i = 1, \cdots, N$$
$$\lambda_{2N-3} = \frac{1}{4}(3y_{i+1} + y_i) - \frac{h}{8}(y'_{i+1} - y'_i) - \frac{h^2}{48}(y''_{i+1} - y''_i),$$
$$\lambda_{2N-2} = y_N$$

Similarities and differences with previous works

- In the past we studied BS methods which are LMMs suited for boundary value problems and also interpretable as spline collocation schemes;
- as well as for BSHOs, a dual QI scheme was derived from BS methods requiring function and first derivative values at the spline breakpoints,

  F. Mazzia and AS, BIT 2009
- the current QI scheme can be considered a generalization of the old one, since it requires Hermite–Birkhoff instead of just Hermite data at the spline breakpoints; same approximation order, $O(h^{D+1})$;
- the coefficients spline computation is now forward (suited for IVPs) (fully or almost); it was fully symmetric in the past (suited for BVPs)
- the spline regularity is now minimal, since equal to the maximal order of the derivative data; it was maximal in the old BS-QI scheme, since equal to the degree minus one.

Test 1

$$1)\ y(x) = e^{-x}\sin(5\pi x)\,,\ x \in [-1\,,\,1]\,,$$

N+1	BSH-QI		BSHO-QI	
	err_∞	order	err_∞	order
16	$2.7 \cdot 10^{-1}$	/	$1.9 \cdot 10^{-2}$	/
32	$3.6 \cdot 10^{-3}$	6.2	$5.0 \cdot 10^{-4}$	5.6
64	$6.0 \cdot 10^{-5}$	5.8	$1.5 \cdot 10^{-5}$	5.2
128	$1.1 \cdot 10^{-6}$	5.8	$4.7 \cdot 10^{-7}$	5.1
256	$2.1 \cdot 10^{-8}$	5.6	$1.5 \cdot 10^{-8}$	5.0
512	$4.5 \cdot 10^{-10}$	5.5	$3.8 \cdot 10^{-10}$	5.5

Spline quasi–interpolants of degree $D = 4$. Uniform mesh.

Test 1

$$1)\ y(x) = e^{-x}\sin(5\pi x)\,,\ x \in [-1\,,\,1]\,,$$

N+1	BSH-QI		BSHO-QI	
	err_∞	order	err_∞	order
16	$1.0 \cdot 10^{-1}$	/	$4.8 \cdot 10^{-4}$	/
32	$4.9 \cdot 10^{-4}$	7.6	$3.6 \cdot 10^{-6}$	7.4
64	$1.6 \cdot 10^{-6}$	8.2	$2.8 \cdot 10^{-8}$	7.2
128	$7.0 \cdot 10^{-9}$	7.8	$2.2 \cdot 10^{-10}$	7.0
256	$2.7 \cdot 10^{-11}$	8.0	$1.8 \cdot 10^{-12}$	7.0
512	$1.1 \cdot 10^{-13}$	7.8	$1.4 \cdot 10^{-14}$	7.1

Spline quasi–interpolants of degree $D = 6$. Uniform mesh.

Test 2

$$2)\ y(x) = 1/(1+25x^2) \qquad x \in [-1\,,\,1]\,,$$

N	BSH-QI		BSHO-QI		BSHO-QIa	
	err_∞	order	err_∞	order	err_∞	order
8	$6.8 \cdot 10^{-2}$	/	$1.4 \cdot 10^{-2}$	/	$3.2 \cdot 10^{-2}$	/
16	$1.8 \cdot 10^{-3}$	5.2	$3.1 \cdot 10^{-4}$	5.6	$1.7 \cdot 10^{-3}$	4.2
32	$1.9 \cdot 10^{-4}$	3.2	$2.8 \cdot 10^{-5}$	3.4	$1.1 \cdot 10^{-4}$	4.0
64	$4.4 \cdot 10^{-6}$	5.5	$7.2 \cdot 10^{-7}$	5.3	$2.3 \cdot 10^{-6}$	5.5
128	$7.2 \cdot 10^{-8}$	5.9	$2.1 \cdot 10^{-8}$	5.1	$4.1 \cdot 10^{-8}$	5.8
256	$1.2 \cdot 10^{-9}$	5.9	$6.2 \cdot 10^{-10}$	5.1	$7.5 \cdot 10^{-10}$	5.7

Spline quasi–interpolants of degree $D = 4$, BSHO-QIa: second derivatives approximated with finite differences of order 4. Uniform mesh.

Variant of the scheme with increased regularity

More regular QI splines associable with the same Hermite-Birkhoff data:

$S_D = D$–degree spline space

B-spline basis for the C^{D-1} case: $B_i, i = -D, \ldots, (N-1)$

- breakpoints at the mesh points: $t_0 < \ldots < t_N$;
- each inner B_i is not vanishing in $D+1$ consecutive mesh cells;
- $\Rightarrow$ simple inner knots in the extended knot vector Θ :

$$\Theta := \{\theta_{-D}, \ldots, \theta_{-1}, \theta_0, \cdots, \theta_N, \theta_{N+1}, \ldots, \theta_{N+D}\}$$

where

$$\begin{array}{llll} \theta_{-D} & = \cdots = \theta_0 := t_0, & \theta_N & = \cdots = \theta_{N+D} := t_N, \\ \theta_n & := t_n, & & n = 1, \ldots, N-1. \end{array}$$

Variant spline coefficients

$$P_k^{(j)}(t_k) = y_k^{(j)}\,, \qquad P_k^{(j)}(t_{k+1}) = y_{k+1}^{(j)}\,, \quad j \neq 1$$

$$P_k^{(1)}(t_k) = y_k^{(1)} + \tau_k/h_k \quad P_k^{(1)}(t_{k+1}) = y_{k+1}^{(1)} + \tau_k/h_k\,.$$

Expressing P_k as a linear combination of B-splines active in $[t_k\,,\,t_{k+1}]$,

$$P_k = (B_{k-D}, \cdots, B_k)\mathbf{c}^{(k)}\,, \quad \mathbf{c}^{(k)} \in \mathbb{R}^{D+1}\,,$$

$$\Updownarrow$$

$$G^{(k)T}(\mathbf{c}^{(k)T}\,,\,\tau_k)^T = \mathbf{w}^{(k)}(y) \quad \text{squared linear system}$$

$$\mathbf{w}^{(k)}(y) := (y_k^{(0)}, y_{k+1}^{(0)}, -h_k y_k^{(1)}, -h_k y_{k+1}^{(1)}, \cdots, -h_k^R y_k^{(R)}, -h_k^R y_{k+1}^{(R)},)^T$$

Variant local coefficient matrix $(D = 2R)$

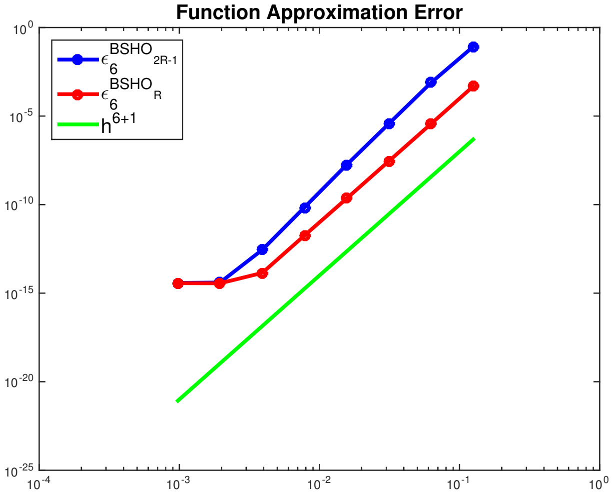
$$G^{(k)} := \left[\begin{array}{ccccc} A_1^{(k)T} & -h_k A_2^{(k)T} & -h_k^2 A_3^{(k)T} & \dots & -h_k^R A_{R+1}^{(k)T} \\ \mathbf{0}^T & \mathbf{e}^T & \mathbf{0}^T & \dots & \mathbf{0}^T \end{array} \right]_{(2R+2)^2}$$

$$A_{j+1}^{(k)} := \left[\begin{array}{ccc} B_{k-D}^{(j)}(t_k), & \dots, & B_k^{(j)}(t_k) \\ B_{k-D}^{(j)}(t_{k+1}), & \dots, & B_k^{(j)}(t_{k+1}) \end{array} \right]_{2\times(2R+1)} \qquad j = 0, \dots, R$$

$\Rightarrow \quad \sigma$ definition in the B-spline basis:

Inner coefficients: $B_i \leftrightarrow \lambda_i(u) =$ R–th entry of $\mathbf{c}^{(n_i)} \leftrightarrow P_{n_i}$
$n_i = i + R$, $i = -R, \dots, N-1-R$ (1 coefficient from each local linear system)

Test1: results for the variant with $D = 2R = 6$



Thanks

THANK YOU FOR YOUR ATTENTION !